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PHYSICALLY NONLINEAR ELLIPSOIDAL INCLUSION IN A LINEARLY ELASTIC MEDIUM

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This paper considers a physically nonlinear ellipsoidal inclusion in an elastic space loaded at infinity by uniform external forces. Relations are obtained that link the stresses and strains at infinite points of the medium and in the inclusion (in the latter, a homogeneous stress–strain state occurs). Some examples, in particular, inclusions in the shape of oblate and prolate spheroids exhibiting nonlinear creep properties, are discussed.

Key words: *physically nonlinear ellipsoidal inclusion, creep, damage parameter, failure.*

Vakulenko and Sevost'yanov [1] proved the following statement, which generalizes classical Eshelby' results [2, 3]: if a linearly elastic space containing a physically nonlinear ellipsoidal inclusion (PNEI) is loaded by uniform external forces at infinity, the stress–strained state (SSS) in the inclusion will be homogeneous. However, concrete relations between the SSS of an elastic medium and the PNEI are not given in [1], although they are easy to determine by comparing the corresponding relations obtained in [1, 2].

1. Linearly Elastic Space with a PNEI. We consider an elastic space with a PNEI loaded at infinity by uniform stresses σ_{kl}^{∞} $(k, l = 1, 2, 3)$. The coordinate system $Ox_1x_2x_3$ is attached to the symmetry axes of the ellipsoid, so that the equation of the boundary Ω separating the inclusion v^* from the elastic region v has the form $x_k^2 a_k^{-2} = 1$ $(a_1 \geqslant a_2 \geqslant a_3)$. Here and below (unless otherwise specified), summation over repeated subscripts is performed from 1 to 3. The strains of the medium and inclusion are considered small; on the surface Ω, the loads and fields are continuous.

In the region v, Hooke's law is valid:

$$\varepsilon_{kl} = a_{klmn}\sigma_{mn}, \qquad \sigma_{kl} = b_{klmn}\varepsilon_{mn} \qquad (k, l = 1, 2, 3), \tag{1.1}$$

It can be written in componentless form

$$\varepsilon = a : \sigma, \qquad \sigma = b : \varepsilon \tag{1.2}$$

(ε_{kl}, σ_{kl}, a_{klmn}, and b_{klmn} are components of the strain, stress, elastic compliance, and elastic modulus tensors) and a and b are mutually inverse tensors.

The constitutive equations for the inclusion v^* are written in general form

$$\varepsilon^* = F(\sigma^*), \qquad \sigma^* = G(\varepsilon^*), \tag{1.3}$$

where F and G are nonlinear tensor operators acting on the stress tensors σ^* and strain tensors ε^*. For (1.3), we also use a componentwise form similar to (1.1).

In [1], the following expressions for the displacement vector components u_k in the region $v^* \cup v$ are obtained:

$$u_k = u_k^{\infty} + \int\limits_{v^*} \Phi_{pq}(\boldsymbol{\xi}) U_{kp,q}(\boldsymbol{r} - \boldsymbol{\xi})\, dv(\boldsymbol{\xi}) \qquad (k = 1, 2, 3), \tag{1.4}$$

$$\boldsymbol{r} = (x_1, x_2, x_3) \in v^* \cup v, \qquad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in v^*$$

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(in [1], the sign ahead of the integral in (1.4) is erroneously replaced by the opposite one), where U is the Green tensor, $\boldsymbol{u}^\infty$ is the displacement vector which is a linear function of $\boldsymbol{r}$ and corresponds to a uniform stress field σ^∞ at infinity, and the subscript q after the comma denotes the derivative with respect to x_q. In the region v^*, the tensor Φ is defined in terms of the quantities introduced in (1.2) and (1.3) as follows:

$$\Phi = \sigma^* - b : \varepsilon^*, \qquad \sigma^* = G(\varepsilon^*). \tag{1.5}$$

It should be noted that Φ, σ^*, and ε^* do not depend on the coordinates x_k $(k = 1, 2, 3)$.

At the same time, for the case where the inclusion undergoes a transformation accompanied by a "free" homogeneous strain ε^T and $\sigma^\infty = 0$, Eshelby [2] obtained relations similar to (1.4):

$$u_k = -\int\limits_{v^*} \sigma^T_{pq} U_{kp,q}(\boldsymbol{r} - \boldsymbol{\xi})\, dv(\boldsymbol{\xi}) \qquad (k = 1, 2, 3). \tag{1.6}$$

Here $\sigma^T = b : \varepsilon^T$. Substituting this equality into (1.6) and differentiating with respect to the coordinates, we find the following strains in the region v^*:

$$\varepsilon^* = S : \varepsilon^T. \tag{1.7}$$

Formula (1.7) can be written in componentwise form

$$\begin{gathered} \varepsilon^*_{kl} = S_{klmn}\varepsilon^T_{mn}, \qquad 2S_{klmn} = -b_{pqmn} \int\limits_{v^*} [U_{kp,ql}(\boldsymbol{r} - \boldsymbol{\xi}) + U_{lp,qk}(\boldsymbol{r} - \boldsymbol{\xi})]\, dv(\boldsymbol{\xi}) \\ (k, l, m, n = 1, 2, 3), \qquad \boldsymbol{r}, \boldsymbol{\xi} \in v^*. \end{gathered} \tag{1.8}$$

As shown in [2], the tensor S does not depend on the coordinates x_k $(k = 1, 2, 3)$ but is determined by the geometry of the region v^* and the elastic characteristics of the ambient medium v. Using (1.2), equality (1.7) can be written as

$$\varepsilon^* = P : \sigma^T, \qquad P = S : a. \tag{1.9}$$

Comparing (1.4) and (1.6), we arrive at the conclusion that in the case of the PNEI considered, to find the strains ε^*_{kl} in the inclusion from formula (1.9), it is necessary to replace σ^T by the tensor $-\Phi$ from (1.5) and to add the strain tensor on infinity ε^∞ corresponding to the first term on the right side of (1.4). Thus, we obtain

$$\varepsilon^* = \varepsilon^\infty - P : (\sigma^* - b : \varepsilon^*)$$

or, in more compact form,

$$\varepsilon^* = \varepsilon^\infty + S : (\varepsilon^* - \tilde{\varepsilon}^*), \qquad \varepsilon^\infty = a : \sigma^\infty, \qquad \tilde{\varepsilon}^* \equiv a : \sigma^*. \tag{1.10}$$

If in v^*, the strains comprise elastic and irreversible strains ε^N_{kl} $(k, l = 1, 2, 3)$, the elastic characteristics of the PNEI and the region v are identical $(\varepsilon^* = a : \sigma^* + \varepsilon^N)$ and $\sigma^\infty = 0$, then relation (1.10) coincides with (1.7), where ε^N plays the role of a free strain tensor.

For the case of an isotropic medium v, the components of the fourth rank tensor S in (1.7)–(1.10) are given in [2], and the displacements of the medium are defined by formulas (1.4) for $\boldsymbol{r} \in v$. The terms on the right side of (1.4) are written in explicit form in [3], where ε^T_{kl} should b replaced by $\varepsilon^*_{kl} - a_{klmn}\sigma^*_{mn}$ $(k, l = 1, 2, 3)$. Using the same replacement, from the dependences given in [2], we obtain the components of the rotation vector in the region v^*.

Relations (1.3) and (1.10) form a closed system from which the SSS of the PNEI, i.e., $\sigma^* = \sigma^*(t)$ and $\varepsilon^* = \varepsilon^*(t)$ (t is time or a loading parameter) is determined from the known loading history at infinity $\sigma^\infty = \sigma^\infty(t)$.

Replacing the elastic constants a_{klmn} $(k, l, m, n = 1, 2, 3)$ in (1.1) and (1.10) by corresponding Volterra operators [4], we obtain the relationships between the SSS in the inclusion and at infinity for the viscoelastic region v with the PNEI.

2. Case of an Isotropic Region $\boldsymbol{v}$. We assume that the elastic medium v is isotropic if relations (1.1) have the form

$$\begin{gathered} E\varepsilon_{kl} = (1 + \nu)\sigma_{kl} - \nu\sigma_{nn}\delta_{kl}, \qquad \sigma_{kl} = 2\mu\varepsilon_{kl} + \lambda\varepsilon_{nn}\delta_{kl} \qquad (k, l = 1, 2, 3), \\ 2\mu = E(1 + \nu)^{-1}, \qquad \lambda = E\nu(1 + \nu)^{-1}(1 - 2\nu)^{-1}, \end{gathered} \tag{2.1}$$

where δ_{kl} are the unit tensor components, E is Young's modulus, ν is Poisson's ratio, and λ and μ are Lamé's constant.

We note that by virtue of (1.10), the relations between $\tilde{\varepsilon}^*_{kl}$ and σ^*_{kl} are similar to relations (2.1) with the same elastic constants.

In this case, the components of the tensor S from (1.8) are defined as follows [2]:

$$S_{kkkk} = Qa_k^2 I_{kk} + RI_k, \qquad S_{kkll} = Qa_l^2 I_{kl} - RI_k,$$

$$2S_{klkl} = 2S_{kllk} = Q(a_k^2 + a_l^2) I_{kl} + R(I_k + I_l),$$

$$Q = 3/[8\pi(1-\nu)], \qquad R = (1-2\nu)/[8\pi(1-\nu)], \tag{2.2}$$

$$I_k = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)\Delta}, \qquad I_{kk} = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)^2\Delta},$$

$$3I_{kl} = 2\pi a_1 a_2 a_3 \int\limits_0^\infty \frac{du}{(a_k^2+u)(a_l^2+u)\Delta}.$$

Here $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$ ($k,l = 1,2,3$; $k \neq l$, no summation over k and l), and the remaining components are $S_{klmn} = 0$.

The quantities I_k, I_{kk}, and I_{kl} from (2.2) are expressed in terms of elliptic integrals of the first and second kind and satisfy the following relations [2]:

$$I_1 + I_2 + I_3 = 4\pi, \qquad I_{k1} + I_{k2} + I_{k3} = 4\pi/(3a_k^2), \qquad a_1^2 I_{k1} + a_2^2 I_{k2} + a_3^2 I_{k3} = I_k,$$

$$I_{kl} = I_{lk} = (I_l - I_k)/[3(a_k^2 - a_l^2)] \qquad (k \neq l, a_k \neq a_l), \tag{2.3}$$

$$3I_{kl} = I_{kk} \qquad (k \neq l, a_k = a_l) \quad (k,l = 1,2,3).$$

From these relations, using known values of I_1 and I_2, we obtain the remaining indicated quantities. The last equality in (2.3) follows from (2.2). In particular, for an oblate spheroid ($a_1 = a_2 = \alpha$, $a_3 = \delta\alpha$, and $\delta < 1$), according to [2], we have

$$I_1 = I_2 = I = 2\pi\delta(1-\delta^2)^{-3/2}[\arccos\delta - \delta(1-\delta^2)^{1/2}]. \tag{2.4}$$

Then, from (2.3) we obtain

$$I_3 = 4\pi - 2I, \qquad I_{11} = I_{22} = 3I_{12} = \frac{3I - 4\pi\delta^2}{4\alpha^2(1-\delta^2)},$$

$$I_{13} = I_{23} = \frac{4\pi - 3I}{3\alpha^2(1-\delta^2)}, \qquad I_{33} = \frac{4\pi(1-3\delta^2) + 6I\delta^2}{3\alpha^2\delta^2(1-\delta^2)}. \tag{2.5}$$

Ignoring the quantity δ^2 compared to unity (i.e., assuming $\delta^2 \ll 1$), from (2.3)–(2.5), we obtain

$$I_1 = I_2 = \pi^2\delta, \qquad I_3 = 4\pi - 2\pi^2\delta, \qquad I_{11} = I_{22} = 3I_{12} = 3\pi^2\delta/(4\alpha^2),$$

$$I_{13} = I_{23} = (4\pi - 3\pi^2\delta)/(3\alpha^2), \qquad \delta^2\alpha^2 I_{33} = 4\pi/3. \tag{2.6}$$

Equality (1.10) is conveniently written in matrix form if the corresponding strain tensor ε is treated as a six-dimensional column vector $\boldsymbol{f}$ with the components $f_1 = \varepsilon_{11}$, $f_2 = \varepsilon_{22}$, $f_3 = \varepsilon_{33}$, $f_4 = \varepsilon_{12}$, $f_5 = \varepsilon_{13}$, and $f_6 = \varepsilon_{23}$ and the tensor S is treated as a 6×6 matrix $\boldsymbol{s}$ whose elements s_{kl} are defined as follows: $s_{kl} = S_{kkll}$ ($k,l = 1,2,3$; no summation over k and l), $s_{44} = 2S_{1212}$, $s_{55} = 2S_{1313}$, $s_{66} = 2S_{2323}$, and the remaining s_{kl} are equal to zero. Then, relation (1.10) becomes

$$f_k^* = f_k^\infty + s_{kl}(f_l^* - \hat{f}_l^*) \qquad (k = 1, 2, \ldots, 6), \tag{2.7}$$

where the summation over l is performed from 1 to 6.

Substituting (2.6) into (2.2), for the elements of the matrix $\boldsymbol{s}$, we obtain

$$s_{11} = s_{22} = (13 - 8\nu)\delta_0, \qquad s_{12} = s_{21} = (8\nu - 1)\delta_0, \qquad s_{13} = s_{23} = 4(2\nu - 1)\delta_0,$$

$$s_{31} = s_{32} = \nu(1-\nu)^{-1} - 4(1+4\nu)\delta_0, \qquad s_{33} = 1 - 8(1-2\nu)\delta_0, \tag{2.8}$$

$$s_{44} = 2(7-8\nu)\delta_0, \qquad s_{55} = s_{66} = 1 - 8(2-\nu)\delta_0, \qquad \delta_0 \equiv \pi\delta/[32(1-\nu)].$$

For a prolate spheroid ($a_1 = \alpha$, $a_2 = a_3 = \delta\alpha$, and $\delta < 1$) according to [2], we have

$$I_2 = I_3 = I = 2\pi\delta^{-1}(\delta^{-2}-1)^{-3/2}[\delta^{-1}(\delta^{-2}-1)^{1/2} - \operatorname{arcosh}\delta^{-1}].$$

Then, from (2.3), we obtain

$$I_1 = 4\pi - 2I, \qquad I_{11} = \frac{4\pi(3-\delta^2) - 6I}{3\alpha^2(1-\delta^2)}, \qquad I_{22} = I_{33} = 3I_{23} = \frac{4\pi - 3I\delta^2}{4\alpha^2\delta^2(1-\delta^2)},$$

$$I_{12} = I_{13} = \frac{3I - 4\pi}{3\alpha^2(1-\delta^2)}. \tag{2.9}$$

Since $I = 2\pi(1 + \delta^2 \ln\delta)$ in the order of δ^2, from (2.9) for $\delta^2 \ll 1$, we obtain (retaining terms of the order of $\delta_1 = -\delta^2 \ln\delta$ and ignoring δ^2 compared to unity)

$$I_1 = 4\pi\delta_1, \qquad I_2 = I_3 = 2\pi(1-\delta_1), \qquad I_{11} = 4\pi\delta_1/\alpha^2,$$

$$\delta^2\alpha^2 I_{22} = \delta^2\alpha^2 I_{33} = 3\delta^2\alpha^2 I_{23} = \pi, \qquad I_{12} = I_{13} = 2\pi(1-3\delta_1)/(3\alpha^2).$$

Substituting these equalities into (2.2), we obtain

$$s_{11} = \frac{2-\nu}{1-\nu}\,\delta_1, \qquad s_{12} = s_{13} = -\frac{1-2\nu}{2(1-\nu)}\,\delta_1, \qquad s_{21} = s_{31} = \frac{\nu - (1+\nu)\delta_1}{2(1-\nu)},$$

$$s_{22} = s_{33} = \frac{5 - 4\nu - 2(1-2\nu)\delta_1}{8(1-\nu)}, \qquad s_{23} = s_{32} = \frac{4\nu - 1 + 2(1-2\nu)\delta_1}{8(1-\nu)}, \tag{2.10}$$

$$s_{44} = s_{55} = \frac{1}{2} - \frac{(1+\nu)\delta_1}{2(1-\nu)}, \qquad s_{66} = \frac{3 - 4\nu - 2(1-2\nu)\delta_1}{4(1-\nu)}, \qquad \delta_1 \equiv -\delta^2 \ln\delta.$$

According to [2], in the case of an elliptic cylinder ($a_3 \to \infty$), we have

$$I_1 = 4\pi a_2(a_1+a_2)^{-1}, \qquad I_2 = 4\pi a_1(a_1+a_2)^{-1}, \qquad I_3 = 0, \qquad I_{12} = 4\pi/[3(a_1+a_2)^2],$$

$$I_{kk} = 4\pi/(3a_k^2) - I_{12} \quad (k = 1, 2), \qquad I_{k3} = 0 \quad (k = 1, 2, 3).$$

In this case, relations (1.10) coincide with those obtained by a different method in [5, 6] for $æ = 3 - 4\nu$, which corresponds to plane deformation.

Below we give formulas for a PNEI which degenerates into:

(a) an elliptic thin plate ($a_1 \geqslant a_2$, $a_3 \to 0$), where $I_k = I_{kl} = 0$ $(k, l = 1, 2)$, $I_3 = 4\pi$, $a_k^2 I_{k3} = 4\pi/3$ ($k = 1, 2, 3$; no summation over k) and the quantities s_{kl} are written in the form of (2.8) for $\delta_0 = 0$;

(b) a needle ($a_1 = \alpha$, $a_2 = a_3 = \delta\alpha$, $\delta \to 0$), where s_{kl} is determined from (2.10) for $\delta_1 = 0$.

Substituting (2.8) for $\delta_0 = 0$ and (2.10) for $\delta_1 = 0$ in (2.7) and taking into account the relations between $\tilde{\varepsilon}^*_{kl}$ and σ^*_{kl} (and the relations between ε^∞_{kl} and σ^∞_{kl}) of the form of (2.1), after simple rearrangements, we obtain:

— for case (a),

$$\varepsilon^*_{kl} = \varepsilon^\infty_{kl} \quad (k, l = 1, 2), \qquad \sigma^*_{k3} = \sigma^\infty_{k3} \quad (k = 1, 2, 3); \tag{2.11}$$

relations similar to (2.11) also hold in the plane problem with an elliptical inclusion degenerating into a slot [6];

— for case (b),

$$\varepsilon^*_{11} = \varepsilon^\infty_{11}, \qquad \varepsilon^*_{1k} + \tilde{\varepsilon}^*_{1k} = 2\varepsilon^\infty_{1k} \quad (k = 2, 3), \qquad (3-4\nu)\tilde{\varepsilon}^*_{23} + \varepsilon^*_{23} = 4(1-\nu)\varepsilon^\infty_{23},$$

$$E(\varepsilon^*_{22} - \varepsilon^*_{33}) = (1+\nu)[4(1-\nu)(\sigma^\infty_{22} - \sigma^\infty_{33}) - (3-4\nu)(\sigma^*_{22} - \sigma^*_{33})], \tag{2.12}$$

$$E(\varepsilon^*_{22} + \varepsilon^*_{33}) = 2(\sigma^\infty_{22} + \sigma^\infty_{33} - \nu\sigma^\infty_{11}) - (1+\nu)(\sigma^*_{22} + \sigma^*_{33}).$$

The last of equalities (2.11) (for $k = 3$) and equalities (2.12) are valid for $\nu \neq 0.5$ and $\nu \to 0.5$ because the right and left sides of the original relations contain the factor $1 - 2\nu$.

3. Some Examples. We consider the case of an isotropic PNEI whose strains comprise elastic and creep strains ε^{*c}_{kl}, so that the original equations (1.3) have the form

$$\varepsilon^* = a^* : \sigma^* + \varepsilon^{*c},$$

where a^* is the elastic compliance tensor, which depends only on two constants E^* and ν^*. For the creep strain rates, we obtain the relations [4]

$$\eta^*_{kl} \equiv \dot\varepsilon^{*c}_{kl} = 3B_1\sigma_i^{*n-1}\sigma^{*0}_{kl}(1-\omega)^{-m}, \qquad \sigma^{*0}_{kl} = \sigma^*_{kl} - (1/3)\sigma^*_{nn}\delta_{kl} \quad (k,l = 1,2,3),$$
$$\dot\omega = B_2\sigma_i^{*p}(1-\omega)^{-m}, \qquad \sigma_i^{*2} = (3/2)\sigma^{*0}_{kl}\sigma^{*0}_{kl}, \tag{3.1}$$

where σ^*_i is the stress intensity, $0 \leqslant \omega \leqslant 1$ is a damage parameter, which is equal to zero in the unstrained state and to unity at the moment of fracture, and B_1, B_2, m, n, and p are positive constants.

Relations (3.1) describe isothermal creep and failure processes for softening materials. In particular, for $\omega \equiv 0$, relations (3.1) correspond to nonlinear viscous flow of undamaged materials.

We assume that at the time $t = 0$, stresses σ^∞_{kl} are applied at infinity and then remain constant. At $t < 0$, the entire region $v^* \cup v$ was in a natural unstrained state; therefore, $\varepsilon^{*c}_{kl}|_{t=0} = 0$ $(k,l = 1,2,3)$ and $\omega|_{t=0} = 0$.

We assume that the elastic characteristics of the medium and the inclusion are identical, i.e., $E^* = E$, $\nu^* = \nu$. (This assumption is not significant because the case $E^* \neq E$ and $\nu^* \neq \nu$ does not involve serious difficulties but only leads to more cumbersome expressions.) Then, as noted above, $\varepsilon^* - \tilde\varepsilon^* = \varepsilon^{*c}$ and since $\varepsilon^{*c}|_{t=0} = 0$, relation (1.10) leads to

$$\sigma^*_{kl}\Big|_{t=0} = \sigma^\infty_{kl} \qquad (k,l = 1,2,3). \tag{3.2}$$

In this case, relation (2.7) becomes

$$f^{*c}_k - s_{kl}f^{*c}_l + \hat f^*_k = f^\infty_k \qquad (k = 1,2,\ldots,6). \tag{3.3}$$

As in Sec. 2, we consider the following inclusions: 1) inclusions in the shape of an oblate spheroid; 2) inclusions in the shape of a prolate spheroid.

1. Let $\sigma^\infty_{11} = \sigma^\infty_{22} = \sigma_0$, $\sigma^\infty_{33} = \sigma_{30}$, and $\sigma^\infty_{kl} = 0$ $(k,l = 1,2,3;\ k \neq l)$. From (3.3) and (2.8) it follows that at $t > 0$, the equalities $\sigma^*_{11} = \sigma^*_{22}$ and $\sigma^*_{kl} = 0$ $(k,l = 1,2,3;\ k \neq l)$ are satisfied. We introduce the designations $\sigma^*_{11} = \sigma^*_{22} = \sigma_1(t)$ and $\sigma^*_{33} = \sigma_3(t)$. Next, from (3.1) we find that $\sigma^*_i = |\sigma_1 - \sigma_3|$, and then

$$\dot f^{*c}_1 = \dot f^{*c}_2 = -\dot f^{*c}_3/2 = F_1(\sigma_1 - \sigma_3)(1-\omega)^{-m}, \qquad \dot\omega = F_2(1-\omega)^{-m}, \tag{3.4}$$

$$F_1 = B_1|\sigma_1 - \sigma_3|^{n-1}, \qquad F_2 = B_2|\sigma_1 - \sigma_3|^p.$$

Substituting (2.1), (2.8), and (3.4) into equalities (3.3) differentiated with respect to t, we obtain

$$\dot\sigma_1 = -AF_1(\sigma_1 - \sigma_3)(1-\omega)^{-m}, \qquad \dot\sigma_3 = BF_1(\sigma_1 - \sigma_3)(1-\omega)^{-m},$$
$$\dot\omega = F_2(1-\omega)^{-m}, \qquad A = E[(1-\nu)^{-1} - 4(8\nu+5)(1+\nu)^{-1}\delta_0], \tag{3.5}$$
$$B = 8E(1+\nu)^{-1}(1-2\nu)\delta_0.$$

The first two equations in (3.5) are obtained by reducing the right and left sides by a factor of $1 - 2\nu$ [as in (2.11) and (2.12)].

By virtue of (3.2), the initial conditions for system (3.5) are written as

$$\sigma_1(0) = \sigma_{10}, \qquad \sigma_3(0) = \sigma_{30}, \qquad \omega(0) = 0. \tag{3.6}$$

[We note that in a more general case, where, for example, $\nu = \nu^*$ but $E \neq E^*$, the left sides of the first and second equations of. (3.5) include the terms $(1 - E/E^*)\{(1 - 12\delta_0)\dot\sigma_1 - [\nu(1-\nu)^{-1} - 4(4\nu+1)\delta_0]\dot\sigma_3\}$ and $8(1 - E/E^*)(1 - 2\nu)\delta_0(\dot\sigma_1 + \dot\sigma_3)$, respectively.]

From Eqs. (3.5) and (3.6), it follows that

$$B\sigma_1 + A\sigma_3 = B\sigma_{10} + A\sigma_{30}. \tag{3.7}$$

Subtracting the second equation of (3.5) from the first equation and converting from t to the new variable $\omega = \omega(t)$, which is an increasing function of t, we have

$$(\sigma_1 - \sigma_3)' = -(A+B)B_1B_2^{-1}\sigma_i^{*n-p-1}(\sigma_1 - \sigma_3), \qquad \sigma_i^* = |\sigma_1 - \sigma_3| \tag{3.8}$$

(the prime denotes differentiation with respect to ω).

Multiplying both sides of (3.8) by $\sigma_1 - \sigma_3$ and taking into account that $(\sigma_1 - \sigma_3)'(\sigma_1 - \sigma_3) = \sigma_i^{*\prime}\sigma_i^*$, for σ_i^* we obtain the following equation (similar to that considered in [7] for the case of instantaneous tension of a rod at $t = 0$ with subsequent conservation of its strain):

$$\sigma_i^{*\prime} + (A+B)B_1B_2^{-1}\sigma_i^{*n-p} = 0. \tag{3.9}$$

From (3.9), we have

$$\begin{gathered} \sigma_i^* = \sigma_{i0}^* f_0(\omega), \qquad f_0(\omega) = [1 - C(p-n+1)\omega]^{1/(p-n+1)}, \\ C(\sigma_{i0}^*) = (A+B)B_1B_2^{-1}\sigma_{i0}^{*n-p-1}, \qquad \sigma_{i0}^* = |\sigma_{10} - \sigma_{30}|. \end{gathered} \tag{3.10}$$

Substituting (3.10) into (3.8) and integrating, we obtain

$$\sigma_1 - \sigma_3 = (\sigma_{10} - \sigma_{30})f_0(\omega). \tag{3.11}$$

From (3.10) and the last equation in (3.5), taking into account the third equality in (3.6), we find t as a function of ω:

$$t = B_2^{-1}\sigma_{i0}^{*-p}\int_0^{\omega}[1 - C(p-n+1)\omega]^{-p/(p-n+1)}(1-\omega)^m\,d\omega,$$

which is the reverse of the function $\omega = \omega(t)$. Knowing the function $\omega = \omega(t)$, from (3.7) and (3.11), we find $\sigma_1 = \sigma_1(t)$ and $\sigma_3 = \sigma_3(t)$.

From (3.9) or (3.10) it follows that with time, i.e., as the value of ω increases, there is a reduction (relaxation) of the stress intensity σ_i^* because $A + B = E[(1-\nu)^{-1} - 12(1+4\nu)(1+\nu)^{-1}\delta_0] > 0$ since $\delta_0 \ll 1$. Is failure of the PNEI, i.e., attainment of the value $\omega = 1$ possible under these conditions? In [7], it is shown that this is possible if $C(p-n+1) < 1$) or $C(p-n+1) = 1$ and $m + 1 - Cp > 0$. In the remaining cases, $t_* \to \infty$ (t_* is the time to failure).

Since $A + B \sim E(1-\nu)^{-1}$, we have $C \sim (1-\nu)^{-1}\varepsilon_*(\sigma_{i0}^*)/\varepsilon^e(\sigma_{i0}^*)$, where $\varepsilon_*(\sigma_{i0}^*) = B_1B_2^{-1}\sigma_{i0}^{*n-p}$ is the creep strain at the moment of failure under uniaxial tension by a stress equal to σ_{i0}^* and $\varepsilon^e(\sigma_{i0}^*) = \sigma_{i0}^*E^{-1}$ is the corresponding elastic strain. For most real media, $\varepsilon_* \geqslant \varepsilon^e$; therefore, $C > 1$, and for viscous materials, $C \gg 1$. Then, from the inequality $C(p-n+1) < 1$, it follows that $p < n + C^{-1} - 1 < n$, which corresponds to friable materials [7]. In this situation, failure of a viscous PNEI for which $p > n$ is impossible.

If, for example, $\sigma_{13}^\infty = \sigma_0$ (or $\sigma_{23}^\infty = \sigma_0$), $\sigma_0 = \text{const}$, $\sigma_0 > 0$, and the remaining σ_{kl}^∞ are equal to zero, then from (3.3) and (2.8) it follows that the single stress component different from zero σ_{kl}^* is the component $\sigma_{13}^* = \sigma_{13}^*(t)$, which satisfies the equation

$$\sigma_{13}^{*\prime} + B_0\sigma_{13}^{*n-p} = 0, \qquad B_0 = 8(2-\nu)(1+\nu)^{-1}EB_1B_2^{-1}3^n\delta_0, \tag{3.12}$$

and

$$\sigma_{13}^*(0) = \sigma_0. \tag{3.13}$$

By analogy with (3.10), from (3.12) and (3.13) we obtain

$$\sigma_{13}^* = \sigma_0[1 - B_0(p-n+1)\sigma_0^{n-p+1}\omega]^{1/(p-n+1)}.$$

In this case, the condition of finiteness of the time t_* becomes

$$B_0(p-n+1)\sigma_0^{n-p-1} < 1, \tag{3.14}$$

which is also possible for a viscous inclusion material for which $p > n$ and $B_3 \equiv EB_1B_2^{-1}\sigma_0^{n-p-1} \gg 1$. As follows from (3.14), in order that these conditions be satisfied, it suffices that the inequalities $0 < p - n < B_0^{-1}\sigma_0^{p-n+1} - 1$ hold, which is the case for $B_0^{-1}\sigma_0^{p-n+1} > 1$, i.e., for $8(2-\nu)(1+\nu)^{-1}3^nB_3\delta_0 < 1$. The last inequality can be satisfied (in spite of the fact that $B_3 \gg 1$) because $\delta_0 \ll 1$.

2. Let $\sigma_{11}^\infty = \sigma_{10}$, $\sigma_{22}^\infty = \sigma_{33}^\infty = \sigma_{20}$, and $\sigma_{kl}^\infty = 0$ $(k,l = 1,2,3;\ k \neq l)$. From (2.3) and (2.10), it follows that at $t > 0$, the following equalities hold: $\sigma_{22}^* = \sigma_{33}^*$ and $\sigma_{kl}^* = 0$ $(k,l = 1,2,3;\ k \neq l)$. We introduce the designations $\sigma_{11}^* = \sigma_1(t)$ and $\sigma_{22}^* = \sigma_{33}^* = \sigma_2(t)$. Next, from (3.1) we find that $\sigma_i = |\sigma_2 - \sigma_1|$ and then

$$\dot{f}_2^{*c} = \dot{f}_3^{*c} = -\dot{f}_1^{*c}/2 = B_1|\sigma_2 - \sigma_1|^{n-1}(\sigma_2 - \sigma_1)(1-\omega)^{-m},$$
$$\dot{\omega} = B_2|\sigma_2 - \sigma_1|^p(1-\omega)^{-m}. \tag{3.15}$$

Substituting (2.1), (2.10), and (3.15) into equalities (3.3) differentiated with respect to ω and reducing by $1 - 2\nu$, we obtain the system

$$(1-\nu^2)\sigma_1' = -E[\nu - 2 - (\nu - 5)\varepsilon]F_3, \qquad (1-\nu^2)\sigma_2' = -E(1-2\nu)(1-2\varepsilon)F_3,$$
$$F_3 = B_1B_2^{-1}|\sigma_2 - \sigma_1|^{n-p-1}(\sigma_2 - \sigma_1)$$

with the initial conditions $\sigma_1(0) = \sigma_{10}$ and $\sigma_2(0) = \sigma_{20}$. Its solution is similar to that given above for (3.5) and (3.6). In particular, the condition of finiteness of the time t_* has the form

$$C_1(p - n + 1) < 1, \qquad C_1 = E(1-\nu^2)^{-1}[3(1-\nu) - (7 - 5\nu)\varepsilon]B_1B_2^{-1}\sigma_{i0}^{*n-p-1},$$
$$\sigma_{i0} = |\sigma_{20} - \sigma_{10}|.$$

From this it follows that $C_1 \sim 3(1+\nu)^{-1}\varepsilon_*(\sigma_{i0}^*)/\varepsilon^e(\sigma_{i0}^*) > 1$ (ε_* and ε^e are defined above); therefore, $p < n + C_1^{-1} - 1 < n$, which corresponds to friable materials.

A similar situation arises when only one of the components σ_{kl}^∞ $(k,l = 1,2,3$ is different from zero; $k \neq l)$. In this case, as follows from (2.10), the corresponding equation of the form (3.12) does not contain a small factor at σ_{kl}^{*n-p}, and, therefore, condition (3.14) (i.e., $t_* < \infty$) cannot be satisfied for the viscous inclusion material.

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